Available online at www.sciencedirect.com
SCIENCE DIRECT®
Chaos, Solitons and Fractals 28 (2006) 1113–1118
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

The golden mean in the topology of four-manifolds, in conformal field theory, in the mathematical probability theory and in Cantorian space-time

L. Marek-Crnjac

University of Maribor, Faculty of Mechanical Engineering, Smetanova ulica 17, SI-2000 Maribor, Slovenia

Accepted 30 August 2005

Abstract

In the present work we show the connections between the topology of four-manifolds, conformal field theory, the mathematical probability theory and Cantorian space-time. In all these different mathematical fields, we find as the main connection the appearance of the golden mean.

1. Introduction

First of all let us draw attention to a close connection between $\varepsilon^{(\infty)}$ theory and the topology of four-manifolds. This connection between Cantorian space-time and iterated capped surface stems from the fact that the grope heights are given by the Fibonacci numbers. A capped surface is a surface in $S \times R^2$ (S a surface) obtained by replacing disks in S with copies of the picture. The disks are called the caps of the surface. The starting surface is a disk and the iterated capped surface constructions define capped gropes. The original surface is the first stage; the surfaces replacing the first stage caps are the second stage and so on. A capped grope has a height at least n if the replacement has been done at least $n-1$ times.

Let a_n denote the height after n steps. The grope heights are then given by [1,2]

$$a_n = 2 + \frac{\sqrt{5}+3}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-3}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The sequence

$$b_n = \frac{\sqrt{5}+3}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-3}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

is the well known Fibonacci sequence. It can be expressed in terms of $\phi = \frac{\sqrt{5}-1}{2}$ and $\frac{1}{\phi} = \frac{1+\sqrt{5}}{2}$ as following:

$$b_n = \frac{\sqrt{5}+3}{2\sqrt{5}}\left(\frac{1}{\phi}\right)^n + \frac{\sqrt{5}-3}{2\sqrt{5}}(-\phi)^n.$$

doi:10.1016/j.chaos.2005.08.160

The Fibonacci sequence b_n can be expressed in other way

$$b_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

and finally the expression of b_n in terms of ϕ and $\frac{1}{\phi}$ can be written as

$$b_n = \frac{\left(\frac{1}{\phi}\right)^n - (-\phi)^n}{\frac{1}{\phi}+\phi}.$$

The grope heights can be now expressed in terms of ϕ as following:

$$a_n = 2 + \frac{\left(\frac{1}{\phi}\right)^n - (-\phi)^n}{\frac{1}{\phi}+\phi}.$$

In the limit these surfaces tend to a fractal-like space akin to $\varepsilon^{(\infty)}$. The Cantorian space-time $\varepsilon^{(\infty)}$ is constructed as an infinite number of elementary Cantor sets with all conceivable Hausdorff dimensions. All possible union and intersections form the $\varepsilon^{(\infty)}$ Cantorian space-time as shown in detail by El Naschie. This space is defined by three dimensions, the formal dimension $d_f = \infty$, the topological dimension $d_T = 4$ and the Hausdorff dimension $\langle d_c \rangle = 4 + \phi^3 = 4.236067977$. The Hausdorff dimension is given in [3] as the expectation of a discrete gamma distribution

$$\langle d_c \rangle = \frac{\sum_{n=0}^{\infty} n^2 \phi^n}{\sum_{n=0}^{\infty} n \phi^n} = \frac{\frac{\phi(1+\phi)}{(1-\phi)^3}}{\frac{\phi}{(1-\phi)^2}} = \frac{1}{\phi^3} = 4.236067977,$$

where $\phi = \frac{\sqrt{5}-1}{2} = 0.618033983$ is the golden mean. The formula for $d_c^{(n)}$ is given in [3] as

$$d_c^{(n)} = \left(\frac{1}{d_c^{(0)}}\right)^{n-1}.$$

When setting $n = 4$ and $d_c^{(0)} = \phi$ we obtain

$$d_c^{(4)} = \frac{1}{\phi^3}.$$

The Hausdorff dimension of $\varepsilon^{(\infty)}$ is now

$$\langle d_c \rangle = d_c^{(4)} = \frac{1}{\phi^3} = 4 + \phi^3 = 4.236067977.$$

The exact topological embedding dimension corresponding to the Hausdorff expectation dimension $\frac{1}{\phi^3} = 4 + \phi^3$ is exactly $d_T = 4$.

In [4] the connection between the geometry of four-manifolds and ϕ is explained. The dimension of the kernel of $\varepsilon^{(\infty)}$ is $\mathrm{Dim\,Ker}\,\varepsilon^{(\infty)} = \phi$. The limit set with the dimension ϕ is equal to the dimension of the kernel of $\varepsilon^{(\infty)}$ and it is a Cantor-like set. Thus the Cokernel of $\varepsilon^{(\infty)}$ has the same dimension as the complement of the limit set

$$\mathrm{Dim\,CoKer}\,\varepsilon^{(\infty)} = 1 - \phi = \phi^2.$$

Following [4]

$$\tau = \mathrm{Dim\,Ker}\,\varepsilon^{(\infty)} - \mathrm{Dim\,CoKer}\,\varepsilon^{(\infty)} = \phi - \phi^2 = \phi(1-\phi) = \phi\phi^2 = \phi^3,$$

and the inverse value of τ is equal to the Hausdorff dimension of $\varepsilon^{(\infty)}$

$$\frac{1}{\tau} = \frac{1}{\phi^3} = 4 + \phi^3.$$

2. Knot theory

Next, we write down some theorems and definitions related to knots in higher dimensions.

Theorem 1. *A locally flat embedding $f: S^2 \to S^4$ is unknotted (isotopic to the standard embedding) if and only if for a homomorphism π_1 is*

$$\pi_1(S^4 - f(S^2)) \approx Z.$$

Following [2] it is shown that for $n \geqslant 2$ an embedding $f: S^n \to S^{n+2}$ is unknotted if and only if the complement is homotopy equivalent to S^1. In higher dimensions there are many knots whose complements have fundamental group Z, but not the homotopy type of S^1.

Definition 2. We call an embedding of S^1 in a homology 3-sphere N "Z-slice" if it extends to a (locally flat) embedding of D^2 in the contractible four-manifold bounding N so that the complement has fundamental group Z, but not the homotopy type of S^1.

Theorem 3. *An embedding $f: S^1 \to N$, N a three-dimensional manifold homology sphere, is Z-slice if and only if the natural homomorphism $\pi_1(N - f(S^1)) \to Z$ has a perfect kernel, or equivalently, the Alexander polynomial of the knot is* 1.

If $f: S^n \to S^{n+2}$ is an embedding then there is a map $(S^{n+2} - f(S^n)) \to S^1$ which is a Z homology equivalence, but when $n > 4$ usually is not a homology equivalence even if $\pi_1(S^{n+2} - f(S^n)) \approx Z$.

This theory does not extend to dimension 4, even for good fundamental groups, these are poly-(finite or cyclic) groups following the Definition 2. In [4] a new view point of how to obtain a real knot in a four-dimensional space is explained. All knots are equivalently trivial and dissolve in this higher dimensional space. A geometrical object corresponding to the circle must have the dimension 2. In a four dimensional space we have therefore a codimension 2. The topological dimension of $\varepsilon^{(\infty)}$ is four. We can have a knot only when the object corresponding to the circle is of the dimension 2 and also the codimension is equal 2. The Frisch–Wasserman–Delbrück conjecture says that the probability for a randomly embedded circle to be knotted tend to one as the length of the circle tends to infinity. For a fractal circle the length is infinite and a fractal circle is everywhere knotted. This is an important mathematical result, which could be used to explain the existence of elementary particles as knots in the fabric of space-time.

3. Polylogarithm identities in a conformal field theory

In conformal field theory it might be surprising to know that we encounter the golden mean. Polylogarithm identities can be expressed in term of $2 - \tau = \phi^2$, where $\tau = \frac{1}{\phi}$, $\phi = \frac{\sqrt{5}-1}{2}$ and also in term of ϕ. The three identities following [5] are given:

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log z \log(1-z),$$
$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{-z}{1-z}\right) = -\frac{1}{2}\log^2(1-z),$$
$$\frac{1}{2}\mathrm{Li}_2(z^2) + \mathrm{Li}_2\left(\frac{-z}{1-z}\right) - \mathrm{Li}_2(-z) = -\frac{1}{2}\log^2(1-z).$$

For the value $2 - \tau = \phi^2 = \frac{3-\sqrt{5}}{2}$ it is possible to express $\mathrm{Li}_2(z)$ in terms of elementary functions as [5]

$$\mathrm{Li}_2(\phi^2) = \frac{\pi^2}{15} - \frac{1}{4}\log^2\phi^2.$$

The dilogarithm $\mathrm{Li}_2(\phi^2)$ can be written in term of ϕ

$$\mathrm{Li}_2(\phi^2) = \frac{\pi^2}{15} - \log\phi.$$

Similar it is possible to obtain $\mathrm{Li}_2(\phi)$ from the first identity by inserting for $z = \phi^2$

$$\mathrm{Li}_2(\phi^2) + \mathrm{Li}_2(1-\phi^2) = \frac{\pi^2}{6} - \log(\phi^2)\log(1-\phi^2),$$

where $1 - \phi^2 = \phi$,

$$\mathrm{Li}_2(\phi) + \mathrm{Li}(\phi) = \frac{\pi^2}{6} - 2\log^2\phi,$$
$$\mathrm{Li}_2(\phi) = \frac{\pi^2}{6} - 2\log^2\phi - \mathrm{Li}_2(\phi^2),$$
$$\mathrm{Li}_2(\phi) = \frac{\pi^2}{10} - \log^2\phi.$$

Inserting $z=\phi^2$ in the second identity we can express $\mathrm{Li}_2(-\phi)$ in term of ϕ

$$\mathrm{Li}_2(\phi^2)+\mathrm{Li}_2\left(\frac{-\phi^2}{1-\phi^2}\right)=-\frac{1}{2}\log^2(\phi),\quad \left(\frac{-\phi^2}{1-\phi^2}\right)=-\phi,$$

$$\mathrm{Li}_2(-\phi)=-\frac{1}{2}\log^2\phi-\mathrm{Li}_2(\phi^2),$$

$$\mathrm{Li}_2(-\phi)=-\frac{1}{2}\log^2\phi-\frac{\pi^2}{15}+\log^2\phi,$$

$$\mathrm{Li}_2(-\phi)=-\frac{\pi^2}{15}+\frac{1}{2}\log^2\phi.$$

Finally we insert $z=\phi$ in the second identity and obtain the expression for $\mathrm{Li}_2\left(-\frac{1}{\phi}\right)$

$$\mathrm{Li}_2(\phi)+\mathrm{Li}_2\left(\frac{-\phi}{1-\phi}\right)=-\log^2\phi,$$

$$\mathrm{Li}_2(\phi)+\mathrm{Li}_2\left(\frac{-\phi}{\phi^2}\right)=-2\log^2\phi,$$

$$\mathrm{Li}_2\left(-\frac{1}{\phi}\right)=-2\log^2\phi-\mathrm{Li}(\phi),$$

$$\mathrm{Li}_2\left(-\frac{1}{\phi}\right)=-2\log^2\phi-\frac{\pi^2}{10}+\log^2\phi,$$

$$\mathrm{Li}_2\left(-\frac{1}{\phi}\right)=-\frac{\pi^2}{10}-\log^2\phi.$$

The dilogarithms $\mathrm{Li}_2(\phi^2)$, $\mathrm{Li}_2(\phi)$, $\mathrm{Li}_2(-\phi)$, $\mathrm{Li}_2\left(-\frac{1}{\phi}\right)$ could be expressed in terms of elementary functions and ϕ.

Polylogarithm identities in $D=3$ are similar. Following [5] one finds

$$\frac{1}{4}\mathrm{Li}_3(z^2)=\mathrm{Li}_3(z)+\mathrm{Li}_3(-z),$$

$$\mathrm{Li}_3(z)+\mathrm{Li}_3\left(\frac{-z}{1-z}\right)+\mathrm{Li}_3(1-z)=\mathrm{Li}_3(1)+\frac{\pi^2}{6}\log(1-z)-\frac{1}{2}\log z\log^2(1-z)+\frac{1}{6}\log^3(1-z).$$

For $z=\phi$ we obtain the following expression:

$$\mathrm{Li}_3(\phi^2)=\frac{4}{5}\mathrm{Li}_3(1)+\frac{\pi^2}{15}\log(\phi^2)-\frac{1}{12}\log^3(\phi^2).$$

The polylogarithm $\mathrm{Li}_3(\phi^2)$ can be written in term of ϕ as

$$\mathrm{Li}_3(\phi^2)=\frac{4}{5}\mathrm{Li}_3(1)+\frac{2\pi^2}{15}\log\phi-\frac{2}{3}\log^3\phi.$$

We see that the properties of the golden mean are of a great importance for the expression of polylogaritms in terms of elementary functions.

4. Mathematical probability theory

The mathematical theory of probability plays a major role in quantum mechanics. Following [6–8] we see that classical probability is involved in the two-slit experiment. The probability to observe a wave like or a particle like behavior is identical and so we can conclude that particles and waves are fundamentally indistinguishable in quantum space-time.

A particle could pass one of the two slits. The probability that the particle goes through slit 1 is P_1 and the probability of going through slit 2 is P_2. Following [6–8] the total probability is given by the union or subtraction of the two events as

$$P=|P_1\pm P_2|$$

where $P_1+P_2=1$.

But a quantum particle can pass through slit one and slit two simultaneously. In this case the intersection rule gives us

$$P=|\pm P_1P_2|.$$

The final result is now the equation of Gödel undecidability [7]

$$|P_1 \pm P_2| = |\pm P_1 P_2|$$

By inserting $P_2 = 1 - P_1$ in the above condition we find

$$P_1^2 + P_1 - 1 = 0,$$

as our first equation.

The solution of this equation is

$$P_1^{(1)} = \phi \quad \text{and} \quad P_1^{(2)} = -\frac{1}{\phi},$$

where $P_2^{(2)} = \phi^2$ and $P_2^{(2)} = \frac{1}{\phi^2}$, $\phi = \frac{\sqrt{5}-1}{2}$.

A second equation which we obtain by inserting $P_2 = 1 - P_1$ is

$$P_1^2 - 3P + 1 = 0.$$

The solution of this quadratic equation is

$$P_1^{(1)} = \phi^2 \quad \text{and} \quad P_1^{(2)} = \frac{1}{\phi^2},$$

where $P_2^{(1)} = -\frac{1}{\phi}$ and $P_2^{(2)} = \phi$.

The total probability is thus

$$|P_1 P_2| = |P_1 - P_2| = |\phi - \phi^2| = \phi^3$$

and for $P_1^{(2)} = -\frac{1}{\phi}$ we obtain

$$|P_1 P_2| = |P_1 - P_2| = \left| -\frac{1}{\phi} \cdot \frac{1}{\phi^2} \right| = \frac{1}{\phi^3} = 4 + \phi^3.$$

We see that the inverse of the total probability of the first solution ϕ^3 is $\frac{1}{\phi^3} = 4 + \phi^3$. This is the expectation value of the Hausdorff dimension of Cantorian space-time $\varepsilon^{(\infty)}$. We should recall that the value of the golden mean $\phi = \frac{\sqrt{5}-1}{2}$ is the Hausdorff dimension of a randomly constructed Cantor set. This is the Mauldin–Williams theorem which states that with the probability one the Hausdorff dimension of a randomly constructed Cantor set is $\phi = \frac{\sqrt{5}-1}{2}$. The Cantorian space-time is consequently a large fuzzy manifold and the hyper Kähler manifold following [7] is what is required by the two-slit experiment as a substructure of $\varepsilon^{(\infty)}$.

The two-slit experiment can be extend to a $3, 4, \ldots$ slit experiment. The solutions of both previous equations are ϕ and ϕ^2 remain the same and the total probability is $4 + \phi^3$ which is the Hausdorff dimension of $\varepsilon^{(\infty)}$

$$\langle d_c \rangle = 4 + \phi^3.$$

This Hausdorff dimension can also be written as

$$\langle d_c \rangle = \sum_{n=1}^{\infty} n\phi^n = 1 \cdot \phi^1 + 2 \cdot \phi^2 + 3 \cdot \phi^3 + \cdots = 4 + \phi^3.$$

In this sum there appears the solutions ϕ and ϕ^2. But the solutions for $3, 4, \ldots$ slit experiments are all elements of the values of ϕ^n following [6]. The equation, which we obtain after rearrangements, should be a quadratic equation of the form

$$P_1^2 + (-1)^{n-1} a_n P_1 + (-1)^n = 0,$$

where $a_1 = 1$, $a_2 = 3$, $a_n = a_{n-2} + a_{n-1}$, $n > 2$.

The solutions of these quadratic equations are the values of ϕ^n.

The value of the golden mean $\phi = \frac{\sqrt{5}-1}{2}$ appears also in another situations. It is conjectured that for three variables X, Y, Z such that $P(X > Y)$, $P(Y > Z)$ and $P(Z > X)$ can all three probabilities be as large as the golden mean ϕ following a classical text book on theory of probability [9].

We can write the sum of three probabilities as

$$P = P_1 + P_2 + P_3 - P_1 P_2 - P_2 P_3 - P_1 P_3 + P_1 P_2 P_3,$$

and $P \leqslant 1$. If $P_1 = P_2 = P_3$, then we can write

$$P = 3P_1 - 3P_1^2 + P_1^3,$$

and for the values $0 < P_1 < \phi$ and $\phi < P_1 < 1$ the probability $P < 1$. For $P_1 = \phi$ the probability P is very close to 1.

5. Conclusion

The golden mean $\phi = \frac{\sqrt{5}-1}{2}$ is fundamental in the Cantorian space-time $\varepsilon^{(\infty)}$. In other mathematical fields such as in mathematical analysis, in conformal field theory, in the topology of four-manifold the golden mean is also of great importance. In the present work we presented the connections between random Cantor set and the two-slit gedanken experiment following [6–8]. It is extremely important to notice that in our discussion of the two-slit experiment we do not use anywhere wave equations of any type. Wave-like behavior is not identical to wave. Also the notion of probability wave is not used or accepted in the present work. Our solution is thus a reinterpretation of Feynman path integral following El Naschie's modification of micro space-time geometry and topology.

References

[1] El Naschie MS. Wild topology, hyperbolic geometry and fusion algebra of high energy particle physics. Chaos, Solitons & Fractals 2002;13:1935–45.
[2] Freedman MH, Quinn F. Topology of 4-manifolds. Princeton, NJ: Princeton University Press; 1990.
[3] El Naschie MS. VAK, vacuum fluctuation and the mass spectrum of high energy particle physics. Chaos, Solitons & Fractals 2003; 17:797–807.
[4] El Naschie MS. Superstrings, knots and noncommutative geometry in $\varepsilon^{(\infty)}$ space. Int J Theor Phys 1998;37(12):2935–51.
[5] Sachdev S. Polylogarithm identities in a conformal field theory in three dimensions. Phys Lett B 1993;309:285.
[6] El Naschie MS. A new solution for the two-slit experiment. Chaos, Solitons & Fractals 2005;25:935–9.
[7] El Naschie MS. The two-slit experiment as the foundation of *E*–infinity of high energy physics. Chaos, Solitons & Fractals 2005;25: 509–14.
[8] El Naschie MS. Stability analysis of the two-slit experiment with quantum particles. Chaos, Solitons & Fractals 2005;26(2):291–4.
[9] Pitman J. Probability. New York: Springer-Verlag; 1993.